$$\langle \hat{H} \rangle = \sum_{\vec{n}} \frac{E_{\vec{n}}}{e^{\beta(E_{\vec{n}}-\mu)}-1} \quad \text{avec } E_{\vec{n}} = \sum_{j=1}^{D} \left(n_j+\tfrac{1}{2}\right)\hbar\omega_j$$

$$= \sum_{j=1}^{D} \frac{1}{2}\hbar\omega_j \underbrace{\sum_{\vec{n}} \frac{1}{e^{\beta(E_{\vec{n}}-\mu)}-1}}_{=N} + \sum_{\vec{n}} \frac{\sum_j n_j \hbar\omega_j}{e^{\beta(\sum_j n_j\hbar\omega_j-\tilde{\mu})}-1}$$

$$= \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j + \int_0^\infty dn_1 \cdots \int_0^\infty dn_D \sum_{j=1}^{D} n_j\hbar\omega_j \sum_{\ell=1}^{\infty} e^{-\ell\beta(\sum_j n_j\hbar\omega_j-\tilde{\mu})}$$

$$= \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j - \int_0^\infty dn_1 \cdots \int_0^\infty dn_D \frac{\hbar}{\beta}\frac{\partial}{\partial\hbar}\left[\sum_{\ell=1}^{\infty}\frac{1}{\ell} e^{-\ell\beta(\sum_j n_j\hbar\omega_j-\tilde{\mu})}\right]$$

$$= \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j - \frac{\hbar}{\beta}\frac{\partial}{\partial\hbar}\left[\sum_{\ell=1}^{\infty}\frac{1}{\ell}e^{\ell\beta\tilde{\mu}}\prod_{j=1}^{D}\int_0^\infty dn_j\, e^{-n_j\ell\beta\hbar\omega_j}\right]$$

$$= \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j - \frac{\hbar}{\beta}\frac{\partial}{\partial\hbar}\left[\frac{1}{(\beta\hbar\bar{\omega})^D}\sum_{\ell=1}^{\infty}\frac{1}{\ell^{D+1}}e^{\ell\beta\tilde{\mu}}\right]$$

$$= \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j + \frac{D}{\beta}\frac{1}{(\beta\hbar\bar{\omega})^D}\sum_{\ell=1}^{\infty}\frac{z^\ell}{\ell^{D+1}}$$

$$= \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j + D k_B T \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D g_{D+1}(z) \quad \text{avec } z = e^{\beta\tilde{\mu}}$$

pour $T > T_c$

$T < T_c$: remplacer N par $N - N_0$ et $g_{D+1}(z)$ par $\zeta(D+1)$

et ajouter $N_0 E_0 = \frac{N_0}{2}\sum_{j=1}^{D}\hbar\omega_j$

$$\Rightarrow \langle \hat{H} \rangle = \frac{N}{2}\sum_{j=1}^{D}\hbar\omega_j + D k_B T\left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D \zeta(D+1)$$

$$\langle \hat{H} \rangle = \frac{N}{2} \sum_{i=1}^{D} \hbar\omega_i + D k_B T \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D \begin{cases} \zeta(D+1) : T < T_c \\ g_{D+1}(z) : T > T_c \end{cases}$$

Chaleur spécifique : $C_V = \frac{1}{N} \frac{\partial \langle \hat{H} \rangle}{\partial T}\Big|_{N,V}$

On utilise $N = \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D g_D(z) = \left(\frac{k_B T_c}{\hbar\bar{\omega}}\right)^D \zeta_D$

$$\iff \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D = \frac{N}{\zeta(D)} \left(\frac{T}{T_c}\right)^D$$

$\underline{T < T_c :}$

$$C_V = \frac{1}{N} (D+1) D k_B T \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D \zeta(D+1) = k_B D (D+1) \frac{\zeta(D+1)}{\zeta(D)} \left(\frac{T}{T_c}\right)^D$$

$\underline{T > T_c :}$

On utilise $0 = \frac{dN}{dT}(T, z(T)) = \left(\frac{D}{T} g_D(z) + g_D'(z) z'(T)\right) \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D$

et $g_D'(z) = \frac{d}{dz} \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^D} = \sum_{\ell=1}^{\infty} \frac{\ell z^{\ell-1}}{\ell^D} = \frac{1}{z} \sum_{\ell=1}^{\infty} \frac{z^\ell}{\ell^{D-1}} = \frac{1}{z} g_{D-1}(z)$

$$\Rightarrow \frac{d}{dT} g_{D+1}(z) = g_{D+1}'(z) z'(T) = \frac{g_D(z)}{z} \left(-\frac{D}{T} \frac{z\, g_D(z)}{g_{D-1}(z)}\right) = -\frac{D}{T} \frac{g_D^2(z)}{g_{D-1}(z)}$$

$$\Rightarrow C_V = \frac{1}{N}(D+1) D k_B \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D g_{D+1}(z) - \frac{1}{N} D k_B T \left(\frac{k_B T}{\hbar\bar{\omega}}\right)^D \frac{D}{T} \frac{g_D^2(z)}{g_{D-1}(z)}$$

$$= k_B D \left[(D+1) \frac{g_{D+1}(z)}{g_D(z)} - D \frac{g_D(z)}{g_{D-1}(z)} \right]$$